THE SMALL PARAMETER METHOD FOR CANONICAL SYSTEMS WITH PERIODIC COEFFICIENTS

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PMM Vol.23, No.1, 1959, pp.15-34

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(Received 10 June 1958)

We consider the system of differential equations

$$\frac{dx}{dt} = [\mathrm{C} + \varepsilon \mathrm{B}(\theta t, \varepsilon)]\, x \qquad (0.1)$$

where C is a constant matrix,

$$\varepsilon \mathrm{B}(\tau, \varepsilon) = \varepsilon \mathrm{B}_1(\tau) + \varepsilon^2 \mathrm{B}_2(\tau) + \ldots$$

$$\mathrm{B}_j(\tau) \in L(0,2\pi), \qquad \mathrm{B}_j(\tau + 2\pi) = \mathrm{B}_j(\tau)$$

almost everywhere,

$$\int_0^{2\pi} \| \mathrm{B}_j(\tau) \| \, d\tau \leqslant \beta_j$$

and series $\epsilon\beta_1 + \epsilon^2\beta_2 + \ldots$ converges for $|\epsilon\phi| < \epsilon_0$.

Systems of a similar form, as well as systems with almost periodic coefficients, have been treated in papers by Chetaev [1], Shtokalo [2], Erugin [3], Malkin [4], Shimanov [5], Cesari [6], Hale [7], Gambill [8], and others. These authors obtained various results.

In this paper we investigate canonical systems of form (0.1); this case has several specific singularities. In this connection, we will be particularly interested in the so-called case of "resonance" which occurs when the eigenvalues of matrix C are congruent (mod $i\Theta$). A whole class of problems of the theory of the dynamic stability of elastic systems [9] is reducible to systems of this type.

As an example, in Section 4 we consider the problem of constructing the region of dynamic instability of "combined" resonance for one equation occurring in the applications.

A major part of the paper, however, is valid for general systems (0.1), not necessarily canonical systems; the approximate integration of system (0.1) is given by formulas (3.17); these formulas have apparently not been noticed previously. They are also valid in the general case.

In the case of the asymptotic stability of general systems the method of Liapunov functions gives an estimate of the value of the small parameter for which there is stability (for example, see [4b], pp. 348-355). In the case of canonical systems, asymptotic stability is impossible and the method of Liapunov functions is inapplicable. The problem of estimating the values of the small parameter which give stability (or instability) for canonical systems seems to be considerably more difficult. It was solved by altogether different methods in papers [10-14b] (see also the survey [13a]). We shall not concern ourselves with this problem here.

To treat canonical systems it is necessary to overcome the following difficulty. The characteristic exponents of (0.1) in the case of stability must be pure imaginary. Expanding them in powers of $\epsilon^{1/p}$, we calculate the coefficients of this expansion. In the final stage of the computation we obtain pure imaginary values as approximate values of the characteristic exponents. It can be proved that the terms which follow and are not calculated could displace these values either to the righ+ or to the left half-plane, and that consequently it is impossible to make any inferences concerning the stability of the system at the end of the calculations.

Theorem 3.1 indicates the cases in which this difficulty can be overcome.

There is in general no theoretical difficulty in obtaining the approximations indicated above. However, as often happens in the applications, there is a great difference between the theoretical and practical possibilities of carrying out the calculations.

Formula (3.17) makes it possible to "integrate" (this term is explained below) system (0.1) without great difficulty up to and including quantities of the order of ϵ^2 (see the example in Section 4). This approximation is often completely satisfactory in practice.

Subsequently we will use the following notation and terminology. The form

$$\langle x, y\rangle = \frac{1}{i}(\mathrm{J}x, y), \qquad \mathrm{J} = \begin{pmatrix} 0 & \mathrm{I}_k \\ -\mathrm{I}_k & 0 \end{pmatrix}$$

defines a nondegenerate indefinite scalar product of zero signature in the $(n-2k)$-dimensional complex space. (I_k is the identity matrix of order k.)

Let W be a matrix (in general, complex). Then

$$\langle Wx, y\rangle = \langle x, W^{+}y\rangle, \text{ where } W^{+} = J^{-1}WJ$$

The matrix W is said to be J-hermitean if $W^{+} = W$; J-skewhermitean, if $W^{+} = -W$; and J-unitary, if $W^{+}W = I_n$.

Every J-hermitean matrix W is of the form $W = JH$, where $H^* = -H$; every J-skewhermitean matrix is of the form $W = JH$, where $H = H^*$; the condition that W be J-unitary may be rewritten in the form $W^*JW = J$; the J-unitary matrices form a group.

In the case of canonical systems the matrix of the coefficients $C + \epsilon B(\theta t, \epsilon)$ is a real J-skewhermitean matrix (the parameters ϵ and U are real). Complex J-skewhermitean matrices might also be considered.

The matrix of the fundamental system of solutions $X(t)$, $X(0) = I_n$, is J-unitary [15]* in the case of canonical systems (the parameters θ and ϵ are fixed).

According to the Liapunov-Poincaré theorem, the characteristic equation

$$\det\left[X\left(\frac{2\pi}{\theta}\right) - \rho I\right] = 0 \tag{0.2}$$

of system (0.1) is recurrent. Its roots $\rho_j(\theta, \epsilon)$ are functions of θ and ϵ.

1. Auxiliary information regarding functions of matrices.

Let G be the union of a finite number of connected and simply connected regions in the complex plane and suppose that $f(z)$ is single-valued and analytic in the interior of G.

Suppose that A is a matrix whose spectrum is in G. The matrix $f(A)$ can be defined as

$$f(A) = \sum_h \int_{\Gamma_h} (\zeta I_n - A)^{-1} f(\zeta)\, d\zeta \tag{1.1}$$

where the Γ_h are non-intersecting circumferences (or other closed curves) contained entirely in G and with the property that each point of the

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* We give a simple proof of this. If $JH(t)$ is the matrix of the coefficients of the canonical system, we have

$$dX/dt = JH(t)X$$

Hence

$$\frac{d}{dt}(X^* JX) \equiv 0, \quad (X^* JX)_t = (X^* JX)_0 = J, \quad \text{или } X^+X = I$$

spectrum of A is contained in the interior of precisely one circumference Γ_h.

This definition implies that $f(S^{-1}AS) = S^{-1}f(A)S$ and if A is decomposed into blocks $A = A_1 + A_2$, then $f(A) = f(A_1) + f(A_2)$. Hence the calculation of $f(A)$ is reduced to the case when A is a Jordan block. In that case, the expression obtained from (1.1) coincides with the usual expression ([15] , p. 132).

Therefore, the definition of $f(A)$ by the Cauchy formula (1.1) is identical with the other possible definitions, those of Smirnov ([16] , pp. 315-338) and Gantmakher ([15] , pp. 83-94).

Lemma 1.1. Let $\phi(z)$ be single-valued and analytic in the region G (which is, in general, not connected), with $\phi(G) = G_1$, and suppose that $\psi(w)$ is single-valued and analytic in the region G_1. Let $f(z) = \psi[\phi(z)]$; $f(z)$ is single-valued and analytic in G. Suppose that the spectrum of matrix A is contained in G [and therefore the spectrum of $B = \phi(A)$ is contained in G_1]. Then $f(A) = \psi(B)$, that is, the two consecutive calculations $B = \phi(A)$ and $\psi(B) = \psi[\phi(A)]$ yield the same result as the single calculation $f(A) = \psi[\phi(A)]$.

Proof. Matrix A and function $\phi(z)$ are known [15, 16] to give rise to a polynomial $\Phi(z) = \Sigma a_m z^m$ such that $\phi(A) \equiv \Sigma a_m A^m$. It suffices to define $\Phi(z)$ by the condition

$$\Phi(\lambda_j) = \varphi(\lambda_j),\quad \Phi'(\lambda_j) = \varphi'(\lambda_j), \ldots,\quad \Phi^{(n-1)}(\lambda_j) = \varphi^{(n-1)}(\lambda_j)$$

where the λ_j are the eigenvalues, and n the order, of A; it would be enough to use instead of n the maximum dimension of the canonical block corresponding to eigenvalue λ_j. The analogous polynomial $\Psi(w)$ is

$$\Psi(\mu_j) = \psi'(\mu_j), \ldots,\quad \Psi^{(n-1)}(\mu_j) = \psi^{(n-1)}(\mu_j)$$

where $\mu_j = \phi(\lambda_j)$ are the eigenvalues of matrix B.

We will show that the polynomial $F(z) \equiv \Psi[\Phi(z)]$ can be used to calculate $f(A)$, i.e. $F(A) = f(A)$. We have

$$\begin{gathered} F(\lambda_j) = \Psi[\Phi(\lambda_j)] = \Psi(\mu_j) = \psi(\mu_j) = f(\lambda_j) \\ F'(\lambda_j) = \Psi'(\mu_j)\,\Phi'(\lambda_j) = \psi'(\mu_j)\,\varphi'(\lambda_j) = f'(\lambda_j) \\ \cdots\cdots\cdots\cdots \\ F^{(n-1)}(\lambda_j) = f^{(n-1)}(\lambda_j) \end{gathered}$$

Hence $F(z)$ and $f(z)$ coincide ([15] , p. 84) on the spectrum of A, i.e. $F(A) = f(A)$.

Let C_1 be the result of calculating $\psi[\phi(A)]$ in two steps, C_2 the result of the single-step calculation. Then

$$B = \varphi(A) = \Phi(A), \qquad C_1 = \psi(B) = \psi(B) = \Psi[\Phi(A)] = F(A)$$

On the other hand, $C_2 = f(A) = F(A)$, i.e. $C_1 = C_2$, as was to be proved.

Lemma 1.2. Suppose that the matrix

$$Y(\varepsilon) = Y_0 + \varepsilon Y_1 + \dots$$

is analytic in ϵ at $\epsilon = 0$, that $Y_0 = \exp A_0$, and that matrix A_0 has no distinct eigenvalues congruent (mod $2\pi i$):

$$\alpha_j - \alpha_h \neq 2\pi m i \qquad (m = \pm 1, \pm 2, \dots) \tag{1.2}$$

Then the function in Y can be defined in a neighborhood of Y_0 so that

(1) $\ln Y(\epsilon)$ is analytic in ϵ at $\epsilon = 0$;

(2) $\ln Y(0) = \ln e A_0 = T_0$.

Proof. Let $\rho_h = e^{\alpha h}$ be the eigenvalues of matrix Y_0. Define $\ln Y$ by

$$\ln Y = \frac{1}{2\pi i} \sum_h \int_{\Gamma_h} (\zeta I - Y)^{-1} (\ln \zeta)_{m_h}\, d\zeta \tag{1.3}$$

Here the Γ_h are circumferences with centers ρ_h and radii small enough for the corresponding circles not to intersect and not to contain the point $\zeta = 0$, and

$$(\ln \zeta)_{m_h} = \ln|\zeta| + i(\arg\zeta + 2\pi \overline{m}_h)$$

is a single-valued branch of $\ln \zeta$. The numbers m_h are chosen to satisfy the condition

$$(\ln \rho_h)_{m_h} = \alpha_h \tag{1.4}$$

This choice is possible only if there is a one-to-one correspondence between the ρ_h and the α_h. This is so here because of (1.2).

The definition of ln Y given above is the same as that which would result from (1.1) for the function

$$\psi(w) = (\ln w)_{m_h}, \quad \text{if} \quad w \in (\Gamma_h)$$

where (Γ_h) is the circle whose circumference is Γ_h

Let $f(z) = \psi(e^z)$ in a neighborhood of the spectrum of matrix A_0.

It follows from (1.4) that $f(z) \equiv z$. By Lemma 1.1, the two-step calculation $Y_0 = e^{A_0}$, $\ln Y_0 = \psi(Y_0)$ yields the same result as the one-step calculation $\ln Y_0 = \psi(e^{A_0}) = f(A_0) = A_0$.

The fact that $\ln Y(\epsilon)$ is analytic in ϵ is a direct consequence of (1.3). This proves Lemma 1.2.

The following remark may have some interest. There is a matrix A_0 [for which (1.2) is not satisfied] such that

(1) if Y is any matrix sufficiently near $Y_0 = e^{A_0}$, there exists a matrix $A = \ln Y$ such that $e^A = Y$;

(2) it is impossible to define $\ln Y$ in a neighborhood of Y_0 so that $\ln Y_0 = \ln eA_0 = T_0$. and so that $\ln Y$ is continuous at Y_0.

It is very easy to construct examples to show this*. For instance, let $A_0 = 2\ 2\pi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then

$$Y_0 = e^{A_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

It is known [15, 16] that the logarithm of an arbitrary matrix sufficiently near I_2 can be defined. Let

$$Q(\mu) = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \qquad (0 < \mu < 1)$$

All the values of $\ln Q(\mu)$ are given by [16.]

$$\ln Q(\mu) = \begin{pmatrix} (\ln \mu)_0 + 2p\pi i & 0 \\ 0 & -(\ln \mu)_0 + 2q\pi i \end{pmatrix} \qquad (p, q = 0, \pm 1, \pm 2, \ldots)$$

Matrix $Q(\mu)$ is arbitrarily near $Y_0 = I_2$ is μ is sufficiently near 1, but matrix $\ln Q(\mu)$ cannot be arbitrarily close to matrix $A_0 = \ln Y_0$.

We now suppose that matrix A_0 is J-skewhermitean. Then, in addition to eigenvalues α_h, matrix A_0 will have eigenvalues $(-\alpha_h)$; and in addition to eigenvalues $\rho_h = e^{\alpha_h}$, matrix Y_0 will have eigenvalues $\rho_h^{-1} = e^{-\alpha_h}$. By (1.4), eigenvalues ρ_h and ρ_h^{-1} will correspond one-to-one to numbers m_h.

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* In general, the matrices Y_0 with multiple eigenvalues for multiple-valued functions play the role of essential singularities: if $Y \to Y_0$, the set of limiting values of $f(Y)$ has the power of the continuum ([16], pp. 332-337). In general, this set consists of a series of "surfaces" and a single isolated point. We eliminate the singularity by assigning to Y, when it is sufficiently near Y_0, a value $f(Y)$ from a neighborhood of this point. This value of $f(Y)$ is said to be regular. These assertions are needed, of course, in more rigorous formulations ([16], pp. 332-337).

Lemma 1.3. Let us suppose that Y and Y_0 are J-unitary matrices and that eigenvalues ρ_h and ρ_h^{-1} of Y_0 correspond one-to-one with numbers m_h of (1.3). Then matrix ln Y defined by (1.3) is J-skewhermitean: $(\ln Y)^+ = -\ln Y$.

Proof. We have

$$(\ln Y)^+ = -\frac{1}{2\pi i}\sum_h \int_{\bar{\Gamma}_h} (\bar{\zeta} I - Y^+)^{-1} (\overline{\ln \zeta})_{m_n} \, d\bar{\zeta}$$

Here $\bar{\Gamma}_h$ denotes the contour conjugate to contour Γ_h (the reflection of contour Γ_h about the real axis). In the last integral we perform an inversion $\bar{\zeta} = \xi^{-1}$. Then

$$\arg \xi = \arg \zeta, \qquad (\overline{\ln \zeta})_m = \ln|\zeta| - i(\arg\zeta + 2\pi m) = -(\ln \xi)_m$$

If Γ'_h is the circumference obtained from Γ_h by the inversion, we get

$$(\ln Y)^+ = -\frac{1}{2\pi i}\sum_h \int_{\Gamma'_h} (\xi^{-1} I - Y^{-1})^{-1} \frac{(\ln \xi)_{m_h}}{\xi^2} \, d\xi$$

(If ζ makes the circuit of Γ_h in the positive direction, ξ makes the circuit of Γ'_h in the negative direction). Since

$$\xi^{-2}(\xi^{-1} I - Y^{-1})^{-1} = \xi^{-1} I + (Y - \xi I)^{-1}, \text{ TO}$$

it follows that

$$(\ln Y)^+ = \frac{1}{2\pi i}\sum_h \int_{\Gamma'_h} [\xi^{-1} I + (Y - \xi I)^{-1}] (\ln \xi)_{m_h} \, d\xi$$

The spectrum of Y_0 is symmetric with respect to the unit circumference; hence there is precisely one (perhaps multiple) eigenvalue of Y_0 in the interior of Γ'_h. Moreover, $\zeta = 0$ is in the exterior of every circumference Γ'_h; hence

$$\int_{\Gamma'_h} \xi^{-1} (\ln \xi)_{m_h} \, d\xi = 0$$

Consequently

$$(\ln Y)^+ = -\frac{1}{2\pi i}\sum_h \int_{\Gamma'_h} (\xi I - Y)^{-1} (\ln \xi)_{m_h} \, d\xi \tag{1.5}$$

The right-hand side of (1.5) differs from the right-hand side of (1.3) in sign and in the order of the terms. But since the center of the circumference Γ'_h is ρ_h^{-1}, and eigenvalues ρ_h and ρ_h^{-1} correspond one-to-one with the numbers m_h by assumption, if we denote by $\rho_{h'}$ the eigenvalue symmetric to ρ_h about the unit circumference: $\rho_{h'} = \rho_h^{-1}$, we see that the integrals over the contours Γ'_h and $\Gamma_{h'}$ are equal.

Hence $(\ln Y)^{+} = -\ln Y$, g.e.d.

2. Reduction of the resonant to the non-resonant case. Setting $\theta t = \tau$ in (0.1), we obtain the equation

$$\frac{dx}{d\tau} = \frac{1}{\theta}[C + \varepsilon B(\tau, \varepsilon)]x \tag{2.1}$$

We shall prove that on our assumptions the matrix of system (1.1) has the form

$$X(\tau, \varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \ldots \tag{2.2}$$

where matrices $X_j(\tau)$ are absolutely continuous and, for sufficiently small ϵ (we consider θ to be fixed in this case), series (2.2) converges uniformly in τ, $0 \leqslant \tau \leqslant \tau_0$.

We give a brief sketch of the usual proof. Substituting the formal series (2.2) into (2.1), we obtain the following recurrence relations for determining the $X_j(\tau)$:

$$\frac{dX_0}{d\tau} = \frac{1}{\theta}CX_0, \qquad \frac{dX_j}{d\tau} = \frac{1}{\theta}[CX_j + (B_1X_{j-1} + \ldots + B_jX_0)]$$

Hence $X_0 = \exp(\tau/\theta\, C)$ and since $X_j(0) = 0$, $j > 1$,

$$X_j = \int_0^{\tau} X_0(\tau - \sigma)[B_1X_{j-1} + \ldots + B_jX_0]_\sigma\, d\sigma$$

Consequently, all the $X_j(\tau)$ are absolutely continuous. Let

$$\xi_j = \max \| X_j(\tau) \| \qquad (j = 0, 1, 2, \ldots;\ 0 \leqslant \tau \leqslant 2\pi)$$

$$\beta_j = \frac{1}{|\theta|}\int_0^{2\pi} \| B_j(\tau) \| \, d\tau \qquad (j = 1, 2, \ldots), \qquad \beta_0 = \frac{2\pi}{|\theta|}\| C \|$$

It is easy to see that $\xi_j \leqslant \eta_j$, where η_j is defined by the recurrence relations

$$\eta_j = e^{\beta_0}(\beta_1\eta_{j-1} + \ldots + \beta_j\eta_0), \qquad \eta_0 = e^{\beta_0} \tag{2.3}$$

Introducing the functions

$$\beta(\varepsilon) = \sum_1^{\infty} \beta_j \varepsilon^j, \qquad \eta(\varepsilon) = \sum_1^{\infty} \eta_j \varepsilon^j$$

we see that (2.3) is equivalent to

$$\eta(\varepsilon) = \frac{\eta_0 \beta(\varepsilon)}{e^{-\beta_0} - \beta(\varepsilon)}$$

Hence both series $\eta_1\epsilon + \eta_2\epsilon^2 + \eta_3\epsilon^3 + \ldots$ and (2.2) converge for $0 \leqslant \tau \leqslant 2\pi$ and for all ϵ, $|\epsilon| < \epsilon_1$, where $\epsilon_1 \leqslant \epsilon_0$ is the least positive root of the equation

$$\exp\left(-\frac{2\pi}{|\theta|}\|C\|\right)=\beta_1\varepsilon+\beta_2\varepsilon^2+\dots$$

Using a general property of systems with periodic coefficients ([17], pp. 179-180), we see that

$$X(\tau+2\pi,\varepsilon)\equiv X(\tau,\varepsilon)\,X(2\pi,\varepsilon)$$

Hence series (2.2) converges uniformly on an arbitrary finite interval $(0, \tau_0)$ for $|\epsilon| < \epsilon_1$. Substitution of (2.2) into the equation

$$X(\tau,\varepsilon)=I+\frac{1}{\theta}\int_0^{\tau}[C+\varepsilon B(\sigma,\varepsilon)]\,X(\sigma,\varepsilon)\,d\sigma$$

yields an identity. Consequently, the matrix of system (2.1) has form (2.2).

By λ_j we denote the eigenvalues of matrix C. In problems of dynamic stability the most interesting case occurs when one of the relations

$$\lambda_j-\lambda_h=im\theta \qquad (m \text{ an integer, } m\neq 0) \tag{2.4}$$

is satisfied.

In this case parametric resonance* is possible and condition (1.2) of Lemma 2.1, which we had intended to apply, is not satisfied. We therefore show first how to reduce this case to that when (2.4) does not hold good.

By $\rho_j^{(0)} = \exp[2\pi\lambda_j/\theta]$ denote the roots of equation (0.2) for $\epsilon = 0$. According to (2.4), there will be multiple roots $\rho_j^{(0)} = \rho_h^{(0)}$ for $\epsilon = 0$. Eigenvalues λ_j are partitioned into residue classes of eigenvalues (mod $i\theta$). Eigenvalues $\lambda_{j_1}, \dots, \lambda_{j_p}$ of a single residue class correspond one-to-one to the roots of the characteristic equation for $\epsilon = 0$:

$$\rho^{(0)}=\exp\left[\frac{2\pi\lambda_{j_1}}{\theta}\right]=\dots=\exp\left[\frac{2\pi\lambda_{j_p}}{\theta}\right]$$

Let $\alpha_0 = (\theta/2\pi)\ln\rho^{(0)}$, where the value of the logarithm is arbitraty. In other words, α_0 is an arbitrary number congruent (mod $i\theta$) with the numbers of the given class. We have

$$\lambda_{j_s}=\alpha_0+im_s\theta \qquad (m_s\text{ — an integer}) \tag{2.5}$$

We define matrix C_0 on the null subspace‡ L_{j_s} corresponding to eigenvalue λ_{j_s} by

* For this to happen, the eigenvalues λ_j must, in addition, be pure imaginary. Here we do not make this assumption.

‡ We recall that the null subspace L corresponding to the eigenvalue λ is the subspace of vectors f for which there exists an integer m such that $(C - \lambda I)^m f = 0$.

$$C_0 f = i m_s f, \qquad f \in L_{j_s} \tag{2.6}$$

Then matrix C_0 is a multiple of the identity matrix on each of the null subspaces of matrix C; hence

$$C_0 C = C C_0 \tag{2.7}$$

Let us put

$$K_0 = \frac{1}{\theta} C - C_0 \tag{2.8}$$

On the invariant subspace L_{j_s} matrix K_0 has the eigenvalue

$$\frac{1}{\theta} \lambda_{j_s} - i m_s = \frac{\alpha_0}{\theta}$$

Hence eigenvalues $\lambda_{j_1}, \ldots, \lambda_{j_s}$ of matrix C belong to a single eigenvalue α_0/θ of matrix K_0^1; the multiplicity of this eigenvalue is equal to the sum of the multiplicities of eigenvalues $\lambda_{j1}, \ldots, \lambda_{js}$. In (2.1) we make the change of variable

$$x = e^{\tau C_0} y$$

and obtain the system

$$\frac{dy}{d\tau} = [K_0 + \varepsilon D(\tau, \varepsilon)]\, y \qquad \left(D(\tau, \varepsilon) = \frac{1}{\theta}\, e^{-\tau C_0} B(\tau, \varepsilon)\, e^{\tau C_0}\right) \tag{2.9}$$

using relation (2.7).

System (2.9), unlike system (2.1), has the property that the class of eigenvalues of matrix K_0 corresponding to one $\rho^{(0)}$ consists of coincident eigenvalues.

It follows from (2.6) that

$$e^{C_0(\tau+2\pi)} = e^{C_0 \tau}$$

on every subspace L_{js} and hence on the whole space.

Therefore $D(\tau, \epsilon)$ is periodic in τ, of period 2π, and analytic in ϵ at $\epsilon = 0$ in the same sense that $B(\tau, \epsilon)$ is, i.e.

$$\varepsilon D(\tau, \varepsilon) = \varepsilon D_1(\tau) + \varepsilon^2 D_2(\tau) + \ldots,$$

where

$$\int_0^{2\pi} \| D_j(\tau) \| \, d\tau \leqslant \delta_j \tag{2.10}$$

and the series $\epsilon\delta_1 + \epsilon^2\delta_2 + \ldots$ converges for $|\epsilon| < \epsilon_0$.

Theorem 2.1. For a suitable choice of α_0 in (2.3):

(a) if system (2.1) is a canonical system (in general, with complex coefficients) with

$$C^{+} = -C, \qquad B(\tau, \varepsilon)^{+} = -B(\tau,\varepsilon) \tag{2.11}$$

then system (2.9) is also a canonical system with

$$K_0^{+} = -K_0, \qquad D(\tau, \varepsilon)^{+} = -D(\tau, \varepsilon) \tag{2.12}$$

(b) if system (2.1) has real coefficients and

$$\theta \neq \pm(2/m)\ \mathrm{Im}\,\lambda_j \qquad (m = 1, 3, 5, \dots)$$

then system (2.9) also has real coefficients;

(c) if (2.1) is a canonical system with real coefficients and

$$\theta \neq \pm(2/m)\,\mathrm{Im}\,\lambda_j \qquad (m = 1, 3, 5, \dots)$$

then (2.9) is also a canonical system with real coefficients.

Proof. (a) We will prove that $C_0^{+} = -C_0$ if $C^{+} = -C$. According to (2.6), if f and g are vectors of the same null subspace L_{j_s}, then

$$\langle C_0 f,\ g\rangle = im_s\langle f,\ g\rangle = -\langle f,\ C_0 g\rangle \tag{2.13}$$

We consider two null subspaces L_j and L_h with eigenvalues λ_j and $\lambda_h = -\lambda_j \neq \lambda_j$. It is easy to verify that eigenvalues λ_j and λ_h cannot belong to the same residue class. If $\lambda_j \equiv a_0 \pmod{i\theta}$, then $\lambda_h \equiv -a_0 \pmod{i\theta}$. Let us agree to choose for the number a_0 of (2.5) the numbers a_0 and $-a_0$ in the classes $\{\lambda_j\}$ and $\{\lambda_h\}$. Then eigenvalues λ_j and $\lambda_h = -\lambda_j$ will correspond to a single number m_s according to (2.5). This verifies (2.13) once more.

We now suppose that $\lambda_h \neq -\lambda_j$ and $\lambda_h \neq \lambda_j$. The corresponding subspace is J-orthogonal ([13], Ch. X). Therefore,

$$\langle C_0 f,\ g\rangle = 0, \qquad \langle f,\ C_0\ g\rangle = 0 \qquad \text{for } f \in L_h,\ g \in L_j$$

i.e., again

$$\langle C_0 f,\ g\rangle = -\langle f,\ C_0 g\rangle \tag{2.14}$$

Hence (2.14) is satisfied for any two vectors f and g of an arbitrary null subspace of matrix C and therefore for two arbitrary vectors. Hence $C_0^{+} = -C_0$.

Then $(e^{C_0 t})^{+} = e^{C_0^{+} t} = e^{-C_0 t}$, i.e. matrix $e^{C_0 t}$ is J-unitary and (2.12) follows from (2.8) and (2.9).

(b) It is enough to prove that matrix C_0 is real for a suitable choice of a_0. We note first that if

$$\overline{C_0 f} = C_0 \bar{f} \tag{2.15}$$

is satisfied for an arbitrary vector f, then matrix C_0 is real. (Here $\bar{f}$ denotes the vector with complex conjugate components.) Indeed, in (2.15) take $f = e_j$, i.e. the column vector all of whose components, except the jth, are equal to zero, and whose jth component is 1. Vector $C_0 e_j$ is the jth column of the matrix C_0. By (2.15), we have $\overline{C_0 e_j} = C_0 \bar{e}_j = C_0 e_j$, i.e. the components of $C_0 e_j$ are real. Consequently, matrix C_0 is real.

Since matrix C is real, the null subspaces L_λ and $L_{\bar\lambda}$ are complex conjugates for a nonreal eigenvalue λ: if $f \in L_\lambda$, then $\bar f \in L_{\bar\lambda}$ and conversely. If by $\bar L$ we denote the subspace consisting of the vectors conjugate to those of L, this may be written as $\bar{L}_\lambda = L_{\bar\lambda}$.

First, suppose that eigenvalues λ_j and $\bar\lambda_j$ belong to the same class and that $\operatorname{Im} \lambda_j \neq 0$. Then

$$\lambda_j - \bar{\lambda}_j = mi\,\theta, \qquad \theta = \frac{2\operatorname{Im} \lambda_j}{m}$$

Because of our assumptions, the number m is even. Since

$$\lambda_j = \operatorname{Re} \lambda_j + {}^1/_2\, im\,\theta,$$

$\lambda_j \equiv \operatorname{Re} \lambda_j \pmod{i\theta}$. Hence we may take $a_0 = \operatorname{Re} \lambda_j$ in (2.5). If λ_{j_s} is in the given class, then $\bar\lambda_{j_s} = a_0 - im_s\theta$ by (2.5), i.e. $\bar\lambda_{j_s}$ also belongs to this class and its corresponding number is $\bar m_s = -m_s$.

Suppose that $f \in L_{\lambda_{j_s}}$. Then $\bar f \in L_{\bar\lambda_{j_s}}$ and according to definition (2.6) of matrix C_0 we have

$$C_0 \bar f = i\bar m_s \bar f = \underline{\underline{}} - im_s \bar f = \overline{C_0 f} \tag{2.16}$$

Hence (2.15) is satisfied for vectors belonging to subspaces of the type indicated above.

If λ_j is a real eigenvalue, set $a_0 = \lambda_j$. The corresponding null space is real, $\bar L_{a_0} = L_{a_0}$ (i.e. L_{a_0} is invariant under the operation of taking the complex conjugate). Since the corresponding number $m_s = 0$, then $C_0 f = 0$, $C_0 \bar f = 0$ for vectors $f \in L_{a_0}$, $\bar f \in L_{a_0}$, i.e. (2.15) is again satisfied.

The only case left to consider is when λ_j and $\bar\lambda_j$ belong to different classes. Then the classes containing λ_j and $\bar\lambda_j$ consist of complex conjugate eigenvalues. If the complex conjugates a_0 and $\bar a_0$ are chosen as the representatives of the classes, eigenvalues λ_{j_s} and $\bar\lambda_{j_s}$ correspond to the numbers m_s and $\bar m_s = -m_s$. Hence (2.16) is satisfied for vectors $f \in L_{j_a}$ and $\bar f \in L_{\bar\lambda_{j_s}}$.

We have shown that (2.15) holds for the f vectors of an arbitrary null subspace of matrix C. Since the whole space is the direct sum of null subspaces, (2.15) is satisfied for an arbitrary vector, i.e. C_0 is a real matrix.

(c) We must show that it is possible to choose the numbers a_0 so that a) and b) are satisfied simultaneously. For this it is necessary that:

(1) a_0 and $-a_0$ be chosen as the representatives of classes $\{\lambda_j\}$ and $\{-\lambda_j\} \neq \{\lambda_j\}$;

(2) the complex conjugates a_0 and a_0 be chosen as the representatives of classes $\{\lambda_j\}$ and $\{\lambda_j\} \neq \{\lambda_j\}$;

(3) if $\{\lambda_j\} = \{\lambda\}$, then $a_0 = a_0$.

It is easy to see that if this is done the set of numbers a_0 will be symmetric with respect to the real and imaginary axes.

Since $C^{+} = -C$ and C is a real matrix, the spectrum of C is symmetric relative to the real and imaginary axes. Consequently, classes $\{\lambda_j\}$, $\{\lambda_j\}$, and $\{-\lambda_j\}$ are identical. It is therefore possible, for classes with $\mathrm{Re}\,\lambda_j \geqslant 0$, to choose numbers a_0 satisfying (2) and (3). In classes with $\mathrm{Re}\,\lambda_j < 0$ we choose the numbers a_0 so that they are symmetric relative to the imaginary axis with the numbers a_0, $\mathrm{Re}\,a_0 \geqslant 0$.

Hence the set of numbers a_0 satisfies (1), (2), and (3).

Note. It is easy to see from the proof of (b) that C_0 is a real matrix if $\theta = m^{-1}\,\mathrm{Im}\,\lambda_j$, $m = \pm 1, \pm 3, \pm 5, \ldots$. Hence (2.9) will be a system with real coefficients if in the corresponding classes the numbers a_0 are chosen as real, and the numbers m_s as multiples of ½. But then matrix $D(\tau, \epsilon)$ will have a period of 4π. If τ is replaced by $2\tau_1$: $\tau = 2\tau_1$, the resulting system will have period 2π and will be of the same form.

The following theorem is a more precise version for our case of the theorem of Liapunov-Floquet on the reduction of a system with periodic coefficients.

Theorem 2.2. Let us assume that a system (2.9) is given, with matrix $D(\tau, \epsilon)$ analytic in ϵ at $\epsilon = 0$ in the sense indicated above and with periodic coefficients of period 2π. We further assume that matrix K_0 does not have distinct eigenvalues congruent (mod i).

We represent matrix $Y(\tau, \epsilon)$ of system (2.9) in the form (2.17)

$$Y(\tau, \varepsilon) = P(\tau, \varepsilon)\, e^{K(\varepsilon)\tau} \tag{2.17}$$

where

$$P(\tau, \varepsilon) = I + \varepsilon P_1(\tau) + \varepsilon^2 P_2(\tau) + \ldots \tag{2.18}$$

$$K(\varepsilon) = K_0 + \varepsilon K_1 + \varepsilon^2 K_2 + \ldots \tag{2.19}$$

are analytic in ϵ at $\epsilon = 0$, $P_j(\tau)$ is an absolutely continuous periodic matrix, with period 2π, and series (2.18) is dominated by a series with constant coefficients. If (2.9) is a canonical system:

then
$$K_0^+ = -K_0, \qquad D(\tau, \varepsilon)^+ = -D(\tau, \varepsilon),$$
$$P(\tau, \varepsilon)^+ P(\tau, \varepsilon) = I, \qquad K(\varepsilon)^+ = -K(\varepsilon) \tag{2.20}$$

The proof is a repetition, using the lemmas of Section 1, of the usual proof of the Liapunov reduction theorem [3a, 17]. It is easy to verify that matrix $Y(\epsilon) = Y(2\pi, \epsilon)$ satisfies the hypotheses of Lemma 1.2; here $A_0 = 2\pi K_0$. Hence we may define $K(\epsilon) = (2\pi)^{-1} \ln Y(2\pi, \epsilon)$ so that $K(\epsilon)$ will be analytic in ϵ at $\epsilon = 0$ and $K(0) = K_0$. Then matrix

$$P(\tau, \varepsilon) = Y(\tau, \varepsilon) e^{-K(\varepsilon)\tau} \tag{2.21}$$

will be analytic in ϵ at $\epsilon = 0$, with coefficients $P_j(\tau)$ which are absolutely continuous functions of τ. It is easily verified that $P(\tau + 2\pi, \epsilon) \equiv P(\tau, \epsilon)$, i.e. $P_j(\tau + 2\pi) = P_j(\tau)$. Since the series for $Y(\tau, \epsilon)$ and $e^{-K(\epsilon)\tau}$ for $0 \leqslant \tau \leqslant 2\pi$ are dominated by a series with constant coefficients, the same is true for series (2.18).

If (2.9) is a canonical system, $Y(\tau, \epsilon)$ is J-unitary. It follows from Lemma 1.3, with $Y = Y(2\pi, \epsilon)$, $Y_0 = Y(2\pi, 0)$, that $K(\epsilon)^+ = -K(\epsilon)$. Since (2.21) implies that $P(\tau, \epsilon)^+ P(\tau, \epsilon) \equiv I$, the theorem follows.

3. Computation of the coefficients of the expansions (2.18) and (2.19). On differentiating (2.17) with respect to τ, we see that $P(\tau, \epsilon)$ satisfies equation

$$\frac{dP}{d\tau} = [K_0 + \varepsilon D(\tau, \varepsilon)] P - PK_0 \tag{3.1}$$

Substituting series (2.18), (2.19), and (2.20) into (3.1), we obtain

$$\frac{dP_n}{d\tau} = K_0 P_n - P_n K_0 + (D_1 P_{n-1} + \ldots + D_{n-1} P_1) - (P_{n-1} K_1 + \ldots + P_1 K_{n-1}) - K_n \tag{3.2}$$

Regarding matrices $P_0 = I$, $P_1, \ldots, P_{n-1}$, $K_1, \ldots, K_{n-1}$ as known, we have equations of the form

$$\frac{dZ}{d\tau} = K_0 Z - ZK_0 + F(\tau) - L \tag{3.3}$$

for determining P_n, K_n.

Here $Z = P_n$, $L = K_n$ and $F(\tau + 2\pi) = F(\tau)$ almost everywhere. The solution $Z(\tau)$ is a periodic matrix: $Z(\tau + 2\pi) = Z(\tau)$. Moreover, $P_n(0) = 0$ for $n \geqslant 1$; hence

$$Z(2\pi) = Z(0) = 0 \tag{3.4}$$

Lemma 3.1. Let us suppose that matrix K_0 in (3.3) (where the unknowns are $Z(\tau)$ and the constant matrix L) does not have distinct eigenvalues congruent (mod i), i.e.

$$\lambda_j - \lambda_h \neq mi, \qquad m = \pm 1, \pm 2, \ldots \qquad \text{and } F(\tau) \in L(0, 2\pi)$$

Then the solution $\{Z(\tau), L\}$, for an arbitrarily given matrix $Z_0 = Z(2\pi) = Z(0)$, exists and is unique, and

$$\begin{gathered} \max_{0 \leqslant \tau \leqslant 2\pi} \|Z(\tau)\| \leqslant \gamma_1 \|Z_0\| + \gamma_2 \int_0^{2\pi} \|F(\tau) - L\| d\tau \\ \|L\| \leqslant \gamma_3 \|Z_0\| + \gamma_4 \int_0^{2\pi} \|F(\tau)\| d\tau \end{gathered} \tag{3.5}$$

The constants γ_1, γ_2, γ_3, γ_4 depend only on matrix K_0. Matrix Z_0 may always be chosen so that

$$L = \frac{1}{2\pi} \int_0^{2\pi} F(\tau)\, d\tau = F_{cp}$$

Then

$$\max_{0 \leqslant \tau \leqslant 2\pi} \|Z(\tau)\| \leqslant \gamma_0 \int_0^{2\pi} \|F(\tau)\| d\tau \tag{3.6}$$

where γ_0 depends only on K_0.

Proof. Treating the matrices as vectors in n^2-dimensional space, we write (3.3) in the form

$$\frac{dZ}{d\tau} = \Lambda Z + F(\tau) - L \tag{3.7}$$

where Λ is the commutator operator: $\Lambda Z = K_0 Z - Z K_0$.

It is known [15] that the numbers $\lambda_j - \lambda_h$, j, $h = 1, \ldots, n$, are the n^2 eigenvalues of operator Λ. At least n of them (when $j = h$) are equal to zero. Denote by Π' the null subspace of operator Λ, corresponding to the zero eigenvalues, and let Π'' be the direct sum of the null subspaces corresponding to the nonzero eigenvalues.

Let Λ' and Λ'' be the operators induced by Λ in the invariant subspaces Π' and Π'' respectively. An arbitrary matrix A can be written as

$$A = A' + A'', \qquad A' \in \Pi', \qquad A'' \in \Pi''$$

Equation (3.7) is split into two equations

$$\frac{dZ'}{d\tau} = \Lambda'Z' + F(\tau)' - L', \qquad \frac{dZ''}{d\tau} = \Lambda''Z'' + F(\tau)'' - L'' \tag{3.8}$$

The condition $Z(2\pi) = Z(0)$ is equivalent to the conditions

$$Z'(2\pi) = Z'(0), \qquad Z''(2\pi) = Z''(0) \tag{3.9}$$

The solution of the second equation of (3.8) is

$$Z''(\tau) = e^{\Lambda''\tau}\left[Z''(0) + \int_0^\tau e^{-\Lambda''\sigma}\left[F(\sigma)'' - L''\right] d\sigma\right] \tag{3.10}$$

Hence the second equality of (3.9) is equivalent to

$$(e^{-2\pi\Lambda''} - I)\, Z''(0) = \int_0^{2\pi} e^{-\Lambda''\sigma} F(\sigma)'' d\sigma - (\Lambda'')^{-1}(e^{-\Lambda'' 2\pi} - I)\, L'' \tag{3.11}$$

Recalling the definition of operator Λ'' we see that the numbers $a_{jh} = \exp[-2\pi(\lambda_j - \lambda_h)] - 1$, for those values of j, h for which $\lambda_j - \lambda_h \neq 0$, are the eigenvalues of operator $e^{-2\pi\Lambda''} - I$.

Because of the assumption that $\lambda_j - \lambda_h \neq \pm mi$, $m = 1, 2, \ldots$, it follows that $a_{jh} \neq 0$. Therefore, the matrix L'' can be uniquely defined for an arbitrary matrix $Z''(0)$, and conversely. In particular, we may take $L'' = F_{cp}$. Equations (3.10) and (3.11) yield the estimate

$$\begin{gathered} \max_{0 \leqslant \tau \leqslant 2\pi} \|Z''(\tau)\| \leqslant \gamma_1'' \|Z''(0)\| + \gamma_2'' \int_0^{2\pi} \|F(\sigma)'' - L\| d\sigma \\ \|L''\| \leqslant \gamma_3'' \|Z''(0)\| + \gamma_4'' \int_0^{2\pi} \|F(\sigma)''\| d\sigma \end{gathered} \tag{3.12}$$

where γ_j'' depends only on matrix K_0. If $L'' = F''_{cp}$, we also get

$$\|Z''(0)\| \leqslant \gamma''_5 \int_0^{2\pi} \|F(\sigma)''\| d\sigma$$

The first equation of (3.8) is split into as many equations as there are blocks in the Jordan canonical form of the matrix of Λ'. In scalar notation every such equation will be a system of the form

$$\frac{d\zeta_1}{d\tau} = \varphi_1 - \varkappa_1, \quad \frac{d\zeta_2}{d\tau} = \zeta_1 + \varphi_2 - \varkappa_2, \ldots, \quad \frac{d\zeta_2}{d\tau} = \zeta_{l-1} + \varphi_l - \varkappa_l \tag{3.13}$$

From these equations, we can find $\zeta_1(\tau), \ldots, \zeta_l(\tau)$ if $\zeta_1(0), \ldots, \zeta_l(0)$ are given. For these functions to be periodic of period 2π it is necessary for the mean values of the right-hand sides to be zero. Hence the solution $\zeta_1(\tau), \ldots, \zeta_l(\tau), \kappa_1, \ldots, \kappa_l$ is uniquely determined for prescribed $\zeta_1(0), \ldots, \zeta_l(0)$.

We may take $\kappa_1 = (\phi_1)_{cp}, \ldots, \kappa_l = (\phi_l)_{cp}$, and choose $\zeta_1(0)$ so that the mean value of the right-hand side of the second equation is zero. $\zeta_2(0), \ldots, \zeta_{l-1}(0)$ are similarly defined, while the value of $\zeta_l(0)$ is left arbitrary. Hence Z_0' can be chosen so that $L' = F'_{cp}$. We will assume that $\zeta_1(0), \ldots, \zeta_l(0)$ are prescribed. It is then easy to see from (3.13) that the numbers κ_j can be estimated by a linear form with positive coefficients in the quantities

$$|\zeta_1(0)|, \ldots |\zeta_{j-1}(0)|, \quad \int_0^{2\pi} |\varphi_1(\sigma)|\, d\sigma, \ldots, \int_0^{2\pi} |\varphi_l(\sigma)|\, d\sigma$$

i.e. that the estimate

$$\|L'\| \leqslant \gamma_3' \|Z(0)'\| + \gamma_4' \int_0^{2\pi} \|F(\sigma)'\|\, d\sigma$$

analogous to the second estimate of (3.12), holds good. The equation and estimate analogous to (3.10) and the first estimate of (3.12) follow from the first equation of (3.8). They are obtained by replacing primes with double primes in the above.

Putting all this together, we get all the assertions of the Lemma, except for estimate (3.6). For $L = F_{cp}$, the preceding argument yields

$$\| Z_0 \| \leqslant \gamma_5 \int_0^{\tau} \| F(\sigma) \|\, d\sigma$$

This, together with (3.5), implies (3.6). This proves the Lemma.

A practical solution $\{Z(\tau), L\}$ is conveniently defined as follows. Let

$$F(\tau) \sim \sum_m F^{(m)} e^{im\tau} \cdot$$

(The series on the right is, in general, divergent, since $F(\tau)$ is only Lebesgue integrable.) According to the above, there exists an absolutely continuous matrix function $Z(\tau)$ which is a solution of (3.3). Let

$$Z(\tau) = \sum_m Z^{(m)} e^{im\tau} \tag{3.14}$$

(The series converges, since $Z(\tau)$ is absolutely continuous.)

Substituting these series into (3.3), we obtain

$$im\, Z^{(m)} = K_0 Z^{(m)} - Z^{(m)} K_0 + F^{(m)} \qquad (m \neq 0), \quad K_0 Z^{(0)} - Z^{(0)} K_0 + F^{(0)} - L = 0 \tag{3.15}$$

because of the uniqueness of the Fourier expansion of a summable function.

Matrices $Z^{(m)}$, $m \neq P$, are (uniquely) defined by the first equation of (3.15). If matrix $Z(0)$ is prescribed, we use it to find

$$Z^{(0)} = Z(0) - \Sigma'_m \qquad \left(\sum_m = \sum_m{}' Z^{(m)}\right)$$

and after that matrix L by means of the second equation of (3.15). Here and subsequently the prime on Σ means that the summation is taken over all $m \neq 0$. If we take $Z(0) = \Sigma'_m$, then $Z^{(0)} = 0$ and $L = F^{(0)} = F_{cp}$.

Remark. This reasoning is almost enough to prove Theorem 3.1, but we have here used the existence of solution $Z(\tau)$ and the convergence of series (3.14). The convergence of series (3.14) and the absolute continuity of $Z(\tau)$ can evidently be deduced from (3.15). (All we know about the $F^{(m)}$ is that they are the Fourier coefficients of a summable matrix function.) This way, however, is hardly shorter.

Hence matrices K_1, $P_1(\tau)$, K_2, $P_2(\tau)$ etc. can be found consecutively by using (3.2).

The condition $P(\tau, \epsilon) \equiv I$ implies that $P_j(0) = 0$, $j = 1, 2, \ldots$. Therefore, $Z(0) = \Sigma_m = 0$ in (3.3), and if we want to use (3.15) we must first determine matrices $Z^{(m)}$, $m \neq 0$ and then matrices $Z^{(0)} = -\Sigma'_m$ and L.

However, it is more convenient to proceed differently. Let $U(\tau, \epsilon)$ be the matrix of a fundamental system of solutions of (2.9) such that

$$U(0, \varepsilon) = V(\varepsilon) = I + \varepsilon V_1 + \varepsilon^2 V_2 + \ldots$$

is analytic in ϵ at $\epsilon = 0$.

Then $V(\epsilon)^{-1}$ will also be analytic in a neighborhood of $\epsilon = 0$ and

$$U(\tau, \varepsilon) = Y(\tau, \varepsilon) V(\varepsilon) = P^{(1)}(\tau, \varepsilon) \exp [K^{(1)}(\varepsilon)\tau]$$

where

$$P^{(1)}(\tau, \varepsilon) = P(\tau, \varepsilon) V(\varepsilon), \qquad K^{(1)}(\varepsilon) = V(\varepsilon)^{-1} K(\varepsilon) V(\varepsilon)$$

Matrices $P = P^{(1)}(\tau, \epsilon)$ and $K = K^{(1)}(\epsilon)$ satisfy (3.1), with $K^{(1)}(0) = K(0) = K_0$. Setting

$$K^{(1)}(\varepsilon) = K_0 + \varepsilon K_1 + \ldots$$
$$P^{(1)}(\tau, \varepsilon) = P_0(\tau) + \varepsilon P_1(\tau) + \varepsilon^2 P_2(\tau) + \ldots$$

we again obtain an equation of the form (3.2) for matrices P_n, K_n. Now, however, $P_n(0) \neq 0$ in general. Moreover, any finite number of matrices $P_n(0) = V_n$ can be chosen arbitrarily; in general, the convergence of the series

$$P^{(1)}(0, \varepsilon) = V(\varepsilon) = I + \varepsilon V_1 + \varepsilon^2 V_n + \dots$$

is sufficient.

It is convenient first to choose $V_n = P_n(0)$ so that

$$K_n = (D_1 P_{n-1} + \dots + D_{n-1} P_1 - P_{n-1} K_1 - \dots - P_1 K_{n-1})_{cp}$$

in (3.2).

According to Lemma 1.3 this can always be done. However, it must be borne in mind that through this choice of matrices K_n we do not obtain matrix $K(\epsilon)$ of (2.17), with $Y(0, \epsilon) \equiv I$, but a matrix similar to it.

Finally, in order not to complicate matters, we assume that

$$B_j(\tau) \in L_2(0, 2\pi)$$

in (0.2).

Then

$$B_j(\tau),\ D_j(\tau) \in L(0, 2\pi), \qquad D_j(\tau) \in L_2(0, 2\pi)$$

If

$$A(\tau),\ B(\tau) \in L_2(0, 2\pi) \qquad A(\tau) \sim \sum_m A_m e^{im\tau}, \quad B(\tau) \sim \sum_m B_m e^{im\tau}$$

then

$$[A(\tau) B(\tau)]_{cp} = \sum_m^{\infty}{}' A_{-m} B_m = 2\mathrm{Re} \sum_{m=0}^{\infty}{}' \bar{A}_m B_m$$

Here the series converges absolutely. Using this formula and (3.15), with $Z^{(0)} = Z_{cp} = 0$, $L = F^{(0)} = F_{cp}$, it is easy to obtain computational formulas for calculating matrices K_1, K_2, K_3, P_1, P_2.

In system (2.9) we represent $D(\tau, \epsilon)$ by series (2.10) and

$$D_j(\tau) \sim \sum_{m=-\infty}^{\infty} D_j^{(m)} e^{im\tau}$$

By $W = K_m(G)$ we denote the solution of equation (3.16)

$$imW = K_0 W - W K_0 + G \tag{3.16}$$

Then

$$K_1 = [D_1(\tau)]_{cp} = D_1^{(0)}, \quad P_1(\tau) = \sum_{m \neq 0} K_m(D_1^{(m)}) e^{im\tau}, \quad K_2 = \sum_{m \neq 0} D_1^{(-m)} K_m(D_1^{(m)})$$

$$F^{(m)} = \sum_{k+l=m} [D_1^{(l)} K_k(D_1^{(k)})] - K_m(D_1^{(m)}) K_1 + D_2^{(m)}$$

$$P_2(\tau) = \sum_{m\neq 0} K_m \cdot (F^{(m)})\, e^{im\tau}$$

$$K_3 = \sum_{m\neq 0} [D_1^{(-m)} K_m (F^{(m)}) + D_2^{(-m)} K_m (D_1^{(m)})] + D_3^{(0)} \tag{3.17}$$

If $D(\tau, \epsilon)$ is a real matrix function, then

$$K_2 = 2\operatorname{Re} \sum_{m=1}^{\infty} D_1^{(m)} K_m (D_1^{(m)}) + D_2^{(0)}$$

$$K_3 = 2\operatorname{Re} \sum_{m=1}^{\infty} [\bar{D}_1^{(m)} K_m (F^{(m)}) + \bar{D}_2^{(m)} K_m (D_1^{(m)})] + D_3^{(0)}$$

It would be easy to write formulas for K_n and $P_n(\tau)$; we do not do so because they are very unwieldy. We will call the expression

$$Y^{(n)}(\tau, \varepsilon) = (I + \varepsilon P_1(\tau) + \dots + \varepsilon^n P_n(\tau)) \exp [(K_0 + \dots + \varepsilon^n K_n)\tau]$$

the approximate solution of nth order of (2.9).

This approximate solution has obvious advantages for the study of the behavior of matrix $Y(\tau, \epsilon)$ as $\tau \to \infty$ in comparison with the approximate solution

$$Y(\tau, \varepsilon) \approx e^{K_0\tau} + \varepsilon Y_1(\tau) + \dots + \varepsilon^n Y_n(\tau)$$

where the right-hand side is a partial sum of the series expansion of $Y(\tau, \epsilon)$ in powers of ϵ.

Let us consider the case of a canonical system in more detail.

We assume that $\lambda_0 = i\omega_0$ is pure imaginary and an m-fold eigenvalue of matrix K_0. Matrix $K(\epsilon)$ has m eigenvalues of the form

$$\lambda_j(\varepsilon) = i\omega_0 + \alpha_j \varepsilon^{p_j/q_j} + \beta_j \varepsilon^{(p_j+1)/q_j} + \dots \tag{3.18}$$

If $\operatorname{Re} \alpha_j > 0$ for some j, then $\operatorname{Re} \lambda_j(\epsilon) > 0$ for all sufficiently small $\epsilon > 0$, and system (0.1) is unstable for sufficiently small $\epsilon > 0$.

If all $\operatorname{Re} \alpha_j < 0$ (for $j = 1, \dots, m$ and for all eigenvalues $i\omega_0$), then all $\operatorname{Re} \lambda_j(\epsilon) < 0$ for $\epsilon \neq 0$, $|\epsilon| \leqslant \epsilon_0$. In this case system (0.1) is asymptotically stable for sufficiently small ϵ. This case, however, cannot occur for canonical systems since such systems cannot be asymptotically stable.

If all $\operatorname{Re} \alpha_j \leqslant 0$ and there are α_j such that $\operatorname{Re} \alpha_j = 0$, then it is necessary to determine coefficients β_j corresponding to these latter values of j (there will be analogous deductions about stability), etc.

In a stable canonical system all the coefficients α_j, β_j, etc. will prove to be pure imaginary.

Theorem 3.1. Suppose that (0.1) is a canonical system and that in the consecutive calculation of coefficients α_j, β_j of expansion (3.18) all these coefficients prove pure imaginary and, at some stage, distinct; so that, if $\lambda_j'(\epsilon)$ is a partial sum of (3.18), then

$$\operatorname{Re}\lambda_j'(\varepsilon)=0, \qquad \lambda_j'(\varepsilon)\neq\lambda_h'(\varepsilon) \qquad (j,\,h=1,\ldots,m;\ j\neq h)$$

for $0<\epsilon<\epsilon_0$.

Then

$$\operatorname{Re}\lambda_j(\varepsilon)=0, \qquad \lambda_j(\varepsilon)\neq\lambda_h(\varepsilon), \qquad (j,\,h=1\ \ldots,m;\ j\neq h)$$

for all sufficiently small $\epsilon>0$.

Proof. Let $\lambda_j''(\epsilon)=\lambda_j(\epsilon)-\lambda_j'(\epsilon)$. As $\epsilon\to 0$,

$$|\lambda_j''(\varepsilon)|=o\left(|\lambda_j'(\varepsilon)-\lambda_h'(\varepsilon)|\right) \qquad (j\neq h) \tag{3.19}$$

Surround the point $i\omega_0$ with a circle whose interior contains no other eigenvalues of matrix K_0. For sufficiently small ϵ this circle will contain only m eigenvalues (3.18). Matrix $K(\epsilon)$ will be J-skewhermitean (Theorem 2.2) and its spectrum will be symmetric with respect to the imaginary axis. Therefore, if the conclusion of the theorem is not satisfied, there are at least two eigenvalues for which

$$\operatorname{Im}\lambda_j(\varepsilon)=\operatorname{Im}\lambda_h(\varepsilon) \qquad (j\neq h)$$

for all sufficiently small ϵ.

Hence

$$\operatorname{Im}[\lambda_j'(\varepsilon)-\lambda_h'(\varepsilon)]=-\operatorname{Im}[\lambda_j''(\varepsilon)-\lambda_h''(\varepsilon)]$$

and since $\operatorname{Re}\lambda_j'(\epsilon)=\operatorname{Re}\lambda_h'(\epsilon)=0$ by assumption, it follows that

$$|\lambda_j'(\varepsilon)-\lambda_h'(\varepsilon)|=|\operatorname{Im}[\lambda_j'(\varepsilon)-\lambda_h'(\varepsilon)]\leqslant|\lambda_j''(\varepsilon)|+|\lambda_h''(\varepsilon)|$$

This contradicts (3.19) and proves the theorem.

We will refer to the case in which matrix $K(\epsilon)$ has pure imaginary eigenvalues, some of which are multiple, for all sufficiently small ϵ, as the singular case. To calculate a finite number of the coefficients α_j, β_j of expansion (3.18), we need to know only a finite number of matrices K_0, K_1, K_2, ... Therefore, in the nonsingular case the determination of only a finite number of matrices K_0, K_1, K_2, ... is required to show whether a canonical system (0.1) is stable or unstable.

In practice it is not necessary to determine the coefficients of expansion (3.18). It is more convenient to proceed as follows.

As we have seen, in the case of a canonical system with real coefficients we can consider matrix $K(\epsilon)$ to be real and J-skewhermitean. The characteristic equation

$$\det [K(\varepsilon) - \lambda I] = 0 \tag{3.20}$$

will therefore have real coefficients and contain only even powers of λ, i.e. it will be of the form*

$$\mu^k + \chi_1 \mu^{k-1} + \ldots + \chi_k = 0 \tag{3.21}$$

where $\mu = \lambda^2$, $2k = n$. Coefficients χ_j are functions of the parameters of the system, and in particular of ϵ and θ. Further, the χ_j are analytic functions of ϵ and $1/\theta$. All solutions of (0.1) will be bounded as $t \to \infty$ (stability) if equation (3.20) has real negative roots μ_j. If some of the roots μ_j are complex or real and positive, then (0.1) has solutions unbounded in t (instability).

Hence the problem of obtaining conditions for the stability (instability) of a canonical system (0.1) is reduced to: (1) the calculation, to prescribed accuracy, of matrix $K(\epsilon)$; (2) the construction of the regions of aperiodic stability for equation (3.21).

The conditions for aperiodic stability (i.e. the conditions that equation (3.21) have real negative roots) are well-known (for instance, see [19], pp. 214-226).

In this connection, it is further necessary to investigate the set (usually a line) on which $\delta = 0$, where δ is the discriminant of (3.21). While $\delta = 0$ on the boundaries of the regions of dynamic instability, it is also true that the line $\delta = 0$ can lie in the regions of stability; in that case the system is stable if matrix $K(\epsilon)$ has canonical blocks and unstable if it does not.

4. Example. Let us consider equation ([9], p. 311)**

$$C_2 \frac{d^2 f}{dt^2} + [I - \varphi_0 A] f = 0 \tag{4.1}$$

* The spectrum of a real J-skewhermitean matrix is symmetric relative to the real and imaginary axes, and is therefore symmetric relative to the origin. Hence the characteristic equation does not change when λ is replaced by $-\lambda$ and so contains only even powers of λ.

** We note, referring to [9], that many problems of the dynamic stability of plates and plane forms of bending reduce to this equation.

where

$$C_2 = \begin{pmatrix} 1/\omega_1^2 & 0 \\ 0 & 1/\omega_2^2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & a_{12} \\ a_{12} & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi_0 = \alpha_0 + \beta_0 \cos \theta t$$

α_0, β_0 are small parameters and $0 < \omega_1 < \omega_2$. Multiplication by C_2^{-1} yields the equation

$$\frac{d^2 f}{dt^2} + [P_0^2 - \psi N] f = 0 \tag{4.2}$$

where

$$P_0 = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \varphi = \alpha + \beta \cos \theta t$$

$$\alpha = \alpha_0 \delta, \qquad \beta = \beta_0 \delta, \qquad \delta = \frac{a_{21}}{\omega_2^2} = \frac{a_{12}}{\omega_1^2}$$

Introducing the notation

$$x_1 = P_0^{1/2} f, \qquad x_2 = P_0^{-1/2} \frac{df}{dt}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \varphi_1 = \frac{\varphi}{\sqrt{\omega_1 \omega_2}}$$

$$C = \begin{pmatrix} 0 & P_0 \\ -P_0 & 0 \end{pmatrix}, \qquad B(\theta t) = \tau_1(\theta t) \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$$

we obtain the system

$$\frac{dx}{dt} = [C + B(\theta t)]\, x \tag{4.3}$$

It is easy to verify that matrices JC and JB(θt), where J =
are symmetric, i.e. that system (4.3) is canonical.

The author has shown [14 b] that for equation (4.2) only the regions corresponding to the "combined" frequency $\theta_0 = \omega_1 + \omega_2$ are "broad" regions of dynamic instability (the tangents to the boundaries at the point (0, θ_0) do not coincide), while all other regions are "narrow" (the tangents to the boundaries of the region at a point on the θ-axis coincide).

We here consider, therefore, the problem of "integrating" equation (4.2) for values of θ near $\theta_0 = \omega_1 + \omega_2$ and determining the corresponding region of dynamic instability. Setting

$$\tau = \theta t, \qquad \gamma = \frac{\theta_0 - \theta}{\theta_0 \theta}$$

we obtain the equation

$$\frac{dx}{d\tau} = \left[\frac{1}{\theta_0} C + \left(\gamma C + \frac{1}{\theta} B(\tau)\right)\right] x \tag{4.4}$$

We think of matrix $\gamma C + (1/\theta) B(\tau)$ as a "perturbation"; we could have prefixed it in equation (4.4) with a small parameter ϵ and then set $\epsilon = 1$ in the final formulas. This corresponds to the fact that the final formulas are true for small α, β, γ.

Omitting the calculations, we write the final result*

$$K_0 = \frac{\omega_2}{\theta_0}\begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix}, \qquad K_1 = \gamma C + \frac{\beta}{2\theta\sqrt{\omega_1\omega_2}}\begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} 0 & \nu_1(I-M) \\ \nu_2(I+M) - \nu_1(3I+M) & 0 \end{pmatrix}$$

where

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \nu_1 = \frac{1}{4\theta^2\theta_0\omega_2}\left[\frac{\alpha^2}{1-2\omega_1/\theta_0} + \frac{\beta^2}{16(1-\omega_1/\theta_0)}\right]$$

$$\nu_2 = \frac{1}{2\theta^2\omega_1\omega_2}\left[\frac{\alpha^2}{1-2\omega_1/\theta_0} + \frac{\beta^2}{8(1-\omega_1/\theta_0)}\right]$$

The characteristic equation Det $(K - \lambda I) = 0$, up to quantities of the second order ($K = K_0 + K_1 + K_2$), has the form

$$\mu^2 + \chi_1\mu + \chi_2 = 0, \qquad \text{где} \quad \mu = \lambda^2$$

$$\chi_1 = \left(\frac{\omega_2}{\theta} - 1 + 2\nu_1\right)^2 + \frac{\omega_1}{\theta}\left(\frac{\omega_1}{\theta} - 2\nu_2 + 4\nu_1\right)$$

$$\chi_2 = \frac{\omega_1}{\theta}\left[\left(\frac{\omega_1}{\theta} - 2\nu_2 + 4\nu_1\right)\left(\frac{\omega_2}{\theta} - 1 + 2\nu_1\right)^2 - \frac{\beta^2}{4\theta^2\omega_1\omega_2}\left(\frac{\omega_2}{\theta} - 1 + 2\nu_1\right)\right]$$

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* The calculation of matrix K_2 is the most laborious. This computation and the determination of the last term in (4.5) was carried out by V.S. Grenkov under the direction of the author, who takes this opportunity to thank V.S. Grenkov for his work. The details of this calculation, as well as the determination of the regions of dynamic instability for the principal resonance $\theta_0 = 2\omega_1/m$, $\theta_0 = 2\omega_2/m$ will be published in Inzhenernyi Sbornik. We note that the calculation of the principal resonance is much simpler, since the fact that equation (4.4) has a solution $x(\tau)$ satisfying the relation $x(\tau + 2\pi)$ can be used. Along the boundaries of the regions of dynamic instability corresponding to the principal resonance, the roots of the characteristic equation (0.2) are fixed; along the boundaries of the regions of dynamic instability corresponding to the combined resonance

$$\theta_0 = (\omega_j + \omega_h)/m, \qquad \omega_j \neq \omega_h, \qquad m = 1,2,\ldots,$$

these roots, as multiple roots, are displaced (in an unknown way) on the unit circumference. This makes the problem of determining the boundaries of dynamic instability of the combined resonance more complicated than that of the principal resonance.

The relation $x(\tau + 2\pi) = (-1)^m x(\tau)$ has been used to construct the regions of the principal resonance by the method of harmonic equilibrium in [9], 57 ff.

The boundaries of the domain of dynamic instability are determined from the equation $\delta \equiv 1/4\,\chi_1^2 - \chi_2 = 0$, and the domain of dynamic instability by the inequality $\delta < 0$.

From the equation $\delta = 0$, for the boundaries of the domain of dynamic stability of the "combined" resonance $\theta = \omega_1 + \omega_2$, we obtain the formula

$$\theta_{\pm} = \theta_0 \pm \frac{\beta}{2\sqrt{\omega_1\omega_2}} - \frac{1}{16\theta_0\omega_1\omega_2}(8\alpha^2 + \beta^2) + \ldots \tag{4.5}$$

Instability occurs for $\theta_- < \theta < \theta_+$.

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Translated by H.K.